Comment on the exterior solutions and their geometry in scalar-tensor theories of gravity

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Abstract

We study series of the stationary solutions with asymptotic flatness properties in the Einstein-Maxwell-free scalar system because they are locally equivalent with the exterior solutions in some class of the scalar-tensor theories of gravity. First, we classify spherical exterior solutions into two types of the solutions, an apparently black hole type solution and an apparently worm hole type solution. The solutions contain three parameters, and we clarify their physical significance. Second, we reduce the field equations for the axisymmetric exterior solutions. We find that the reduced equations are partially the same as the Ernst equations. As simple examples, we derive new series of the static, axisymmetric exterior solutions, which correspond to Voorhees's solutions. We then show a non-trivial relation between the spherical exterior solutions and our new solutions. Finally, since null geodesics have conformally invariant properties, we study the local geometry of the exterior solutions by using the optical scalar equations and find some anomalous behaviors of the null geodesics.

1 Introduction

Recently, as natural alternatives to general relativity, the scalar-tensor theories have been studied by many theoretical physicists. In these theories, the gravity is mediated not only by a tensor field but also by a scalar field. Also, such theories have been of interest as effective theories of the string theory at low energy scales [1].

Several theoretical predictions in the scalar-tensor theories have been obtained (see e.g. Ref.[2]∼[5]). It has been found that a wide class of the scalar-tensor theories can pass all the experimental tests in weak gravitational fields. However, it has also been found that the scalar-tensor theories show different aspects of the gravity in the strong gravitational fields in contrast to general relativity (see e.g. Ref.[5]). It has been shown numerically that nonperturbative effects in the scalar-tensor theories increase the maximum mass of an isolated system such as a neutron star [2-5]. In these works, numerical methods play an important role, and, in interpreting the numerical results, a static, spherically symmetric vacuum solution (a spherical exterior solution) is matched to the numerical solution.

The spherical exterior solution is an analytic exact solution, and, as far as the authors know, only few exact solutions have been known. In particular, the axisymmetric exterior

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solution corresponding to the Kerr solution must be of significant interest. Motivated by this fact, we study series of the stationary solutions with asymptotic flatness properties in the Einstein-Maxwell-free scalar system because, as summarized in Appendix A, they are locally equivalent with the exterior solutions in some class of the scalar-tensor theories of gravity. The field equations are given by

$$\begin{aligned} R_{\mu\nu} &= 8\pi T_{\mu\nu} + 2\varphi_\mu\varphi_\nu, \\ \Box\varphi &= 0, \end{aligned} \tag{1.1}$$

where $T_{\mu\nu}$ is a energy-momentum tensor of the Maxwell field (see also Appendix A).

In this paper we first classify spherical exterior solutions into two types of the solutions, an apparently black hole type (ABH) solution and an apparently worm hole type (AWH) solution. We then clarify physical significance of the parameters contained in the solutions and show that the ABH solution and the AWH solution correspond to, respectively, the mass dominant case and the charge dominant case of the Reissner-Nordström solution. Second, we reduce the field equations for the axisymmetric exterior solutions and show that the reduced equations are partially the same as the Ernst equations. As simple examples, we derive new series of static, axisymmetric exterior solutions, which will be referred to as scalar-tensor-Weyl solutions. We then show a non-trivial relation between the spherical exterior solutions and our new solutions. Finally, we study the local geometry of the exterior solutions by using the optical scalar equations and find some anomalous behaviors of the null geodesics.

2 Spherical exterior solutions

2.1 Classification of the exterior solutions

In this section, we derive static, spherically symmetric exterior solutions with the electric field, which are hereafter referred to as spherical exterior solutions. In the Just coordinate, the metric becomes [4]:

$$ds^2 = -\mathrm{e}^{\gamma} dt^2 + \mathrm{e}^{-\gamma} d\chi^2 + \mathrm{e}^{\lambda-\gamma} d\Omega^2, \tag{2.1}$$

where γ and λ are functions of χ. Non-vanishing components of the electromagnetic tensor, $F_{\mu\nu}$, are $F_{01} = -F_{10} = E(\chi)$. The Maxwell equation is reduced to

$$(E\mathrm{e}^{\lambda-\gamma})' = 0, \tag{2.2}$$

where a prime denotes a derivative with respect to χ. With (2.2) one obtains

$$E = Q\mathrm{e}^{\gamma-\lambda}, \tag{2.3}$$

where the integration constant, Q, is an electric charge.

The Einstein equation and the field equation for φ become

$$\frac{1}{2}(\gamma'' + \lambda'\gamma')\mathrm{e}^{\gamma} = Q^2\mathrm{e}^{2(\gamma-\lambda)}, \tag{2.4a}$$

$$\frac{1}{2}(\gamma'' - 2\lambda'' - \lambda'^2 - \gamma'^2 + \lambda'\gamma')\mathrm{e}^{\gamma} = 2\varphi'^2\mathrm{e}^{\gamma} - Q^2\mathrm{e}^{2(\gamma-\lambda)}, \tag{2.4b}$$

$$\mathrm{e}^{\gamma-\lambda} + \frac{1}{2}(\gamma'' - \lambda'' + \lambda'\gamma' - \lambda'^2)\mathrm{e}^{\gamma} = Q^2\mathrm{e}^{2(\gamma-\lambda)}, \tag{2.4c}$$

$$(\varphi'' + \lambda'\varphi')\mathrm{e}^{\gamma} = 0. \tag{2.4d}$$

With (2.4a) and (2.4c), one obtains

$$(\mathrm{e}^{\lambda})'' = 2. \tag{2.5}$$

One finds two types of the solutions to (2.5), which are referred to as an *apparently black hole type* (ABH) solution and an *apparently worm hole type* (AWH) solution, respectively:

$$\begin{array}{llll} \text{ABH} & \mathrm{e}^{\lambda} = \chi^2 - a\chi & (\chi > a \geq 0), & (2.6\mathrm{a}) \\ \text{AWH} & \mathrm{e}^{\lambda} = \chi^2 + \dfrac{a^2}{4} & (-\infty < \chi < \infty), & (2.6\mathrm{b}) \end{array}$$

where a is an integration constant.

When $a \neq 0$, the equation (2.4d) is integrated as

$$\begin{array}{lll} \text{ABH} & \varphi = \varphi_0 + \dfrac{c}{a} \ln\left(1 - \dfrac{a}{\chi}\right), & (2.7\mathrm{a}) \\ \text{AWH} & \varphi = \varphi_0 + \dfrac{2c}{a}\left[\arctan\left(\dfrac{2\chi}{a}\right) - \dfrac{\pi}{2}\right], & (2.7\mathrm{b}) \end{array}$$

where φ_0 and c are integration constants [2-5].

The equation (2.4a) is integrated as

$$(\gamma' \mathrm{e}^{\lambda})^2 = 4Q^2 \mathrm{e}^{\gamma} + \epsilon b^2, \tag{2.8}$$

where b is an integration constant, and ϵ is 1 in the ABH solution and -1 in the AWH solution.

With (2.4a), (2.4b) and (2.4c), we obtain

$$\frac{\lambda'^2 - \gamma'^2}{4} - \mathrm{e}^{-\lambda} = \varphi'^2 - Q^2 \mathrm{e}^{\gamma - 2\lambda}, \tag{2.9}$$

which is reduced to the following relation among the integration constants:

$$\begin{array}{lll} \text{ABH} & a^2 - b^2 = 4c^2 \longrightarrow a \geq b \geq 0, & (2.10\mathrm{a}) \\ \text{AWH} & b^2 - a^2 = 4c^2 \longrightarrow b \geq a \geq 0. & (2.10\mathrm{b}) \end{array}$$

Since $\mathrm{e}^{\gamma} \longrightarrow 1$ as $\chi \longrightarrow \infty$ in the asymptotically flat spacetime, with (2.8), we obtain the following inequality for the AWH solution:

$$b^2 \leq 4Q^2. \tag{2.11}$$

That is, the AWH solution is *charge dominant*, and no AWH solution exists when $Q = 0$.

2.2 The ABH solution

The metric function, γ, in the ABH solution is obtained as follows:

$$\mathrm{e}^{\gamma} = \frac{B}{(1-B)^2}\left(\frac{b}{Q}\right)^2, \tag{2.12a}$$

where

$$B \equiv k\left(1 - \frac{a}{\chi}\right)^{\frac{b}{a}}, \quad k \equiv \frac{\sqrt{4Q^2 + b^2} - b}{\sqrt{4Q^2 + b^2} + b} < 1. \tag{2.12b}$$

In the limit, $Q \to 0$, the solution coincides with the previously known solution [2-5]. In the limit, $b \to 0$, the solution is reduced to the following form:

$$\mathrm{e}^{\gamma} = \left[1 - \frac{Q}{a} \ln\left(1 - \frac{a}{\chi}\right)\right]^{-2}. \tag{2.13}$$

When $a = b$, i.e., $c = 0$, the Just coordinate, χ, is related to the Schwarzschild coordinate, r, as

$$\chi = r - \frac{kb}{1-k}. \tag{2.14}$$

The metric function, γ, becomes

$$\mathrm{e}^{\gamma} = \left[1 - \left(\frac{kb}{1-k}\right)\frac{1}{r}\right]\left[1 - \left(\frac{b}{1-k}\right)\frac{1}{r}\right] \equiv \left(1 - \frac{r_-}{r}\right)\left(1 - \frac{r_+}{r}\right) \equiv 1 - \frac{2m}{r} + \frac{Q^2}{r^2}, \tag{2.15a}$$

where

$$r_+ \equiv \frac{b}{1-k}, \quad r_- \equiv \frac{kb}{1-k}, \quad 2m \equiv \frac{1+k}{1-k}b = \sqrt{4Q^2 + b^2}. \tag{2.15b}$$

With (2.15b), we obtain the following inequality:

$$0 \leq b^2 = 4(m^2 - Q^2). \tag{2.16}$$

That is, the ABH solution corresponds to the *mass dominant* ($m^2 > Q^2$) Reissner-Nordström spacetime and has two event horizons, r_+ and r_-, when $a = b$.

When $a > b$, the Just coordinate, χ, is related to the Schwarzschild coordinate, r, as

$$r^2 = \chi^2 \left(1 - \frac{a}{\chi}\right) \frac{(1-B)^2}{B} \left(\frac{Q}{b}\right)^2, \tag{2.17}$$

which vanishes at $\chi = a$, 0 and $\chi_B \equiv a/\left(1 - k^{-a/b}\right) < 0$. Moreover, we find that $\chi = a$, $\chi = 0$ and $\chi = \chi_B$, correspond to, respectively, $r = r_+$, $r = r_-$ and $r = 0$ when $a = b$ and that the null surface, $\chi = a$, becomes a singularity when $a > b$.

2.3 The AWH solution

When $a > 0$, the metric function, γ, in the AWH solution is obtained as

$$\mathrm{e}^{\gamma} = \frac{1}{4}\left(\frac{b}{Q}\right)^2 \sec^2\left[\frac{b}{a}\arctan\left(\frac{2\chi}{a}\right) + \beta\right], \tag{2.18a}$$

where the constant, β, is defined by

$$\sec^2\left[\frac{\pi b}{2a} + \beta\right] = \frac{4Q^2}{b^2}. \tag{2.18b}$$

As mentioned previously, the AWH solution does not exist when $Q = 0$.

When $a = 0$, the AWH solution becomes

$$\begin{aligned} e^{\lambda} &= \chi^2, && (2.19a) \\ \varphi &= \varphi_0 - \frac{c}{\chi}, && (2.19b) \\ e^{\gamma} &= \frac{c^2}{Q^2} \sec^2\left(\beta - \frac{c}{\chi}\right), && (2.19c) \\ & && (2.19d) \end{aligned}$$

where

$$\sec^2\beta = \frac{Q^2}{c^2}. \tag{2.20}$$

When $a = b$, the AWH solution is reduced to the following:

$$\begin{aligned} e^{\lambda} &= \chi^2 + \frac{b^2}{4}, && (2.21a) \\ e^{\gamma} &= \frac{1}{4}\left(\frac{b}{Q}\right)^2 \sec^2\left[\arctan\left(\frac{2\chi}{b}\right) + \beta\right], && (2.21b) \\ \varphi &= \varphi_0, && (2.21c) \end{aligned}$$

where

$$\sec^2\left(\frac{\pi}{2} + \beta\right) = \frac{4Q^2}{b^2}. \tag{2.22}$$

Note that (2.22) is equivalent with the following:

$$\sin\beta = \pm\frac{b}{2Q}, \quad \cos\beta = \sqrt{1 - \frac{b^2}{4Q^2}}. \tag{2.23}$$

Then (2.21b) becomes

$$e^{\gamma} = \frac{b^2}{4Q^2}\left(1 + 4\frac{\chi^2}{b^2}\right)\left(\sqrt{1 - \frac{b^2}{4Q^2}} \mp \frac{\chi}{Q}\right)^{-2}, \tag{2.24}$$

and the Just coordinate, χ, is related to the Schwarzschild coordinate, r, as

$$r = e^{(\lambda-\gamma)/2} = \chi \mp \sqrt{Q^2 - \frac{b^2}{4}} \equiv r^{\mp}. \tag{2.25}$$

When we take r^-, we have

$$e^{\gamma} = 1 + 2\sqrt{Q^2 - \frac{b^2}{4}}\frac{1}{r} + \frac{Q^2}{r^2} \equiv 1 - \frac{2m}{r} + \frac{Q^2}{r^2}, \quad m \equiv -\sqrt{Q^2 - \frac{b^2}{4}} < 0, \tag{2.26}$$

which is a negative mass solution and should be discarded as an unphysical exterior solution. When we take r^+, we have

$$e^{\gamma} = 1 - 2\sqrt{Q^2 - \frac{b^2}{4}}\frac{1}{r} + \frac{Q^2}{r^2} \equiv 1 - \frac{2m}{r} + \frac{Q^2}{r^2}, \quad 0 < m \equiv \sqrt{Q^2 - \frac{b^2}{4}}. \tag{2.27}$$

That is, the AWH solution corresponds to the *charge dominant* ($m^2 < Q^2$) Reissner-Nordström spacetime and has a naked timelike singularity, $r = 0$, when $a = b$.

When $a < b$, the Just coordinate, χ, is related to the Schwarzschild coordinate, r, as

$$r^2 = 4\left(\frac{Q}{b}\right)^2 \left(\chi^2 + \frac{a^2}{4}\right) \cos^2\left[\frac{b}{a}\arctan\left(\frac{2\chi}{a}\right) + \beta\right]. \tag{2.28}$$

Note that $r = 0$ at $\chi = \chi_{\mathrm{W}}$, where

$$\chi_{\mathrm{W}} = \frac{a}{2}\tan\left[\frac{a}{b}\left(\frac{\pi}{2} - \beta\right)\right]. \tag{2.29}$$

The AWH solution has a naked timelike singularity at $\chi = \chi_{\mathrm{W}}$. That is, the AWH solution does not have a worm hole, and the asymptotic region defined by $\chi \to -\infty$ does not exist.

3 The Ernst equations in scalar-tensor theories

In the stationary, axisymmetric spacetime, there exist two Killing vectors, $\boldsymbol{\xi} = \partial_t$ and $\boldsymbol{\eta} = \partial_\phi$. It is shown that the energy-momentum tensor, $T^{\alpha\beta}$, of the Maxwell field satisfies the following relations [6]:

$$\xi^\alpha T_\alpha{}^{[\beta}\xi^\gamma\eta^{\delta]} = \eta^\alpha T_\alpha{}^{[\beta}\xi^\gamma\eta^{\delta]} = 0. \tag{3.1}$$

Moreover, one finds that the similar relations hold for $T^{[\varphi]}_{\mu\nu} \equiv 2\partial_\mu\varphi\partial_\nu\varphi$. Accordingly, with the theorem 7.1.1 in Ref.[7], the metric can be reduced to the following form:

$$ds^2 = -\mathrm{e}^{2\psi}(dt - \omega d\phi)^2 + \mathrm{e}^{-2\psi}\left[\mathrm{e}^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2\right], \tag{3.2}$$

where the metric functions, ψ, γ and ω, are functions of $x^1 \equiv \rho$ and $x^2 \equiv z$.

After long and complicated calculations, we find that the field equations are reduced to the following:

$$\mathrm{e}^{2\psi} = \frac{1}{2}\left(\mathcal{E} + \bar{\mathcal{E}}\right) + \Phi\bar{\Phi}, \tag{3.3a}$$

$$\mathrm{e}^{2\psi}\nabla^2\mathcal{E} = (\nabla\mathcal{E})\cdot\left[(\nabla\mathcal{E}) + 2\bar{\Phi}\nabla\Phi\right], \tag{3.3b}$$

$$e^{2\psi}\nabla^2\Phi = (\nabla\Phi)\cdot\left[(\nabla\mathcal{E}) + 2\bar{\Phi}\nabla\Phi\right], \tag{3.3c}$$

$$\begin{aligned}\gamma_{,1} = {} & \rho\left[(\psi_{,1})^2 - (\psi_{,2})^2\right] - \frac{1}{4\rho}\left[(\omega_{,1})^2 - (\omega_{,2})^2\right]\mathrm{e}^{4\psi} \\ & -\rho(\Phi_{,1}\bar{\Phi}_{,1} - \Phi_{,2}\bar{\Phi}_{,2})\mathrm{e}^{-2\psi} + \rho\left[(\varphi_{,1})^2 - (\varphi_{,2})^2\right],\end{aligned} \tag{3.3d}$$

$$\gamma_{,2} = 2\rho\psi_{,1}\psi_{,2} - \frac{1}{2\rho}\omega_{,1}\omega_{,2}\mathrm{e}^{4\psi} - \rho(\Phi_{,1}\bar{\Phi}_{,2} + \Phi_{,2}\bar{\Phi}_{,1})\mathrm{e}^{-2\psi} + 2\rho\varphi_{,1}\varphi_{,2}, \tag{3.3e}$$

$$\nabla^2\varphi = 0, \tag{3.3f}$$

where, for any functions, f and h,

$$\nabla^2 f \equiv \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{\partial^2 f}{\partial z^2}, \quad \nabla f\cdot\nabla h \equiv \frac{\partial f}{\partial\rho}\frac{\partial h}{\partial\rho} + \frac{\partial f}{\partial z}\frac{\partial h}{\partial z}, \quad f_{,i} \equiv \frac{\partial f}{\partial x^i}. \tag{3.4}$$

In Ref.[8], one will find the explicit forms of the metric function, ω, and the Maxwell field in terms of the complex potentials, $\mathcal{E}$ and Φ. Note that the first, second and third equations are the same as those in general relativity and are referred to as *the Ernst equations* of the Einstein-Maxwell system. The remaining equations contain the scalar field contributions, and, as for the metric functions, effects of the scalar field only appear in γ.

4 Scalar-tensor-Weyl solutions.

4.1 The reduced Ernst equation

In this section, we shall consider static, axisymmetric vacuum solutions. In this case, $\omega = \Phi = 0$, and the field equations (3.3a)$\sim$(3.3f) are reduced to the following:

$$\mathrm{e}^{2\psi} = \mathcal{E}, \tag{4.1a}$$

$$\mathcal{E}\nabla^2\mathcal{E} = (\nabla\mathcal{E})\cdot(\nabla\mathcal{E}), \tag{4.1b}$$

$$\gamma_{,1} = \rho\left[(\psi_{,1})^2 - (\psi_{,2})^2\right] + \rho\left[(\varphi_{,1})^2 - (\varphi_{,2})^2\right], \tag{4.1c}$$

$$\gamma_{,2} = 2\rho\psi_{,1}\psi_{,2} + 2\rho\varphi_{,1}\varphi_{,2}, \tag{4.1d}$$

$$\nabla^2\varphi = 0, \tag{4.1e}$$

where $\mathcal{E}$ is a real function. One finds that the Ernst equation (4.1b) is reduced to

$$\nabla^2\psi = 0. \tag{4.2}$$

That is, φ and ψ are harmonic functions.

We introduce *oblate* and *prolate* coordinates, (x, y), defined by

$$\rho = \sigma\sqrt{(x^2+\epsilon)(1-y^2)}, \tag{4.3a}$$

$$z = \sigma x y, \tag{4.3b}$$

where σ is a positive constant, and $\epsilon = \pm 1$. The cases, $\epsilon = 1$ and $\epsilon = -1$, are referred to as *oblate* and *prolate*, respectively. The Ernst equation is then reduced to

$$\frac{\partial}{\partial x}\left[(x^2+\epsilon)\frac{\partial\psi}{\partial x}\right] + \frac{\partial}{\partial y}\left[(1-y^2)\frac{\partial\psi}{\partial y}\right] = 0. \tag{4.4}$$

The similar equation holds for φ.

4.2 Prolate solutions

The simplest prolate solution for ψ is given by

$$\psi = \frac{\delta}{2}\ln\left(\frac{x-1}{x+1}\right), \tag{4.5}$$

where δ is an integration constant. The similar solution for φ is given by

$$\varphi = \varphi_0 + \frac{d}{2}\ln\left(\frac{x-1}{x+1}\right), \tag{4.6}$$

where φ_0 and d are integration constants. Then the corresponding solution for γ becomes

$$e^{2\gamma} = \left(\frac{x^2-1}{x^2-y^2}\right)^{\Delta^2}, \tag{4.7}$$

where

$$\Delta^2 \equiv \delta^2 + d^2. \tag{4.8}$$

For completeness, an explicit form of the metric is shown below:

$$\begin{aligned} ds^2 &= -\left(\frac{x-1}{x+1}\right)^{\delta} dt^2 + \sigma^2 \left(\frac{x-1}{x+1}\right)^{-\delta} \times \\ &\left[\left(\frac{x^2-1}{x^2-y^2}\right)^{\Delta^2} (x^2-y^2)\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right) + (x^2-1)(1-y^2)d\phi^2\right]. \end{aligned} \tag{4.9}$$

Note that, though ψ and φ are y-independent, γ depends on both x and y. The metric contains two parameters, δ and Δ. Now we show that the metric is reduced to the previously known one when we take specific values of the parameters. When $\Delta = \delta$, i.e., $d = 0$, the metric is reduced to one of the Weyl series of the solutions in general relativity (Voorhees's prolate solution [9]). Therefore, we refer to (4.9) as the scalar-tensor-Weyl solution. Voorhees's prolate solution contains the Schwarzschild solution as a specific case, $\delta = 1$, and the Schwarzschild coordinates are related to (x, y) as

$$x = \frac{r}{m} - 1, \quad y = \cos\theta, \quad \sigma = m, \tag{4.10}$$

where m is a usual mass parameter.

The case, $\Delta = 1$, is the most interesting one. We introduce new parameters, a, b and c, and new coordinates (the Just coordinates) by

$$d = \frac{2c}{a}, \quad \sigma = \frac{a}{2}, \quad \delta = \frac{b}{a}, \tag{4.11}$$

and

$$x = \frac{2\chi}{a} - 1, \quad y = \cos\theta. \tag{4.12}$$

Then we find that the solution coincides with the spherical exterior solution with a vanishing electric charge:

$$ds^2 = -\left(1-\frac{a}{\chi}\right)^{\frac{b}{a}} dt^2 + \left(1-\frac{a}{\chi}\right)^{-\frac{b}{a}} d\chi^2 + \chi^2\left(1-\frac{a}{\chi}\right)^{\frac{a-b}{a}} \left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{4.13}$$

Moreover, when $\delta = 1$, i.e., $a = b$, the solution is reduced to Schwarzschild's one.

The right hand sides in (4.1c) and (4.1d) are in the forms of the superposition of the contributions of ψ and φ. It is important to note that the superposition of the *effectively non-spherical* ψ-terms corresponding to Voorhees's prolate solution with $\delta \neq 1$ and the *spherical* configuration of φ reduces to the solution containing the spherical exterior solution. When the spherical scalar field configuration is added to the ψ-terms corresponding to Voorhees's prolate solution with $\delta = 1$, namely the Schwarzschild solution, we cannot have spherical solutions unless $a = b$. The relation among these solutions is schematically shown in Figure 1. The

Figure 1: A schematic picture of series of the scalar-tensor-Weyl solutions.

scalar-tensor-Weyl solution generically has a singularity, $x = 1$, whose topology is S^2, and qualitative features of the singularity may be similar to those in Voorhees's prolate solution. When $\Delta = 1$, the singularity, $x = 1$, becomes a point, which is given $\chi = a$ in the Just coordinate. Moreover, when $\Delta = \delta = 1$, the topology of $\chi = a$, corresponding to the event horizon, $r = 2m$, becomes S^2.

4.3 Oblate solutions

The simplest oblate solution for ψ is given by

$$\psi = \delta \cdot \mathrm{arccot} x, \tag{4.14}$$

where δ is an integration constant. The similar solution for φ is given by

$$\varphi = \varphi_0 + d \cdot \mathrm{arccot} x, \tag{4.15}$$

where φ_0 and d are integration constants. Then the corresponding solution for γ becomes

$$\mathrm{e}^{2\gamma} = \left(\frac{x^2 + y^2}{x^2 + 1} \right)^{\Delta^2}, \tag{4.16}$$

where $\Delta^2 = \delta^2 + d^2$. For completeness, an explicit form of the metric is shown below:

$$\begin{aligned} ds^2 \quad = \quad & -\mathrm{e}^{2\delta \cdot \mathrm{arccot} x} dt^2 + \sigma^2 \mathrm{e}^{-2\delta \cdot \mathrm{arccot} x} \times \\ & \left[\left(\frac{x^2 + y^2}{x^2 + 1} \right)^{\Delta^2} (x^2 + y^2) \left(\frac{dx^2}{x^2 + 1} + \frac{dy^2}{1 - y^2} \right) + (x^2 + 1)(1 - y^2) d\phi^2 \right]. \end{aligned} \tag{4.17}$$

When $\Delta = \delta$, the metric is reduced to Voorhees's oblate solution. In contrast to the prolate case, the oblate solution is not reduced to the spherical exterior solutions for any δ and Δ.

5 Local behaviors of null geodesics

5.1 The spherical exterior solution

In this section, we examine local behaviors of irrotational null geodesics in the spherical exterior solution with a vanishing electric charge. The metric is rewritten in the Schwarzschild coordinate as

$$
\begin{aligned}
&ds^2 = -\left(1-\frac{a}{\chi(r)}\right)^{\frac{b}{a}} dt^2 + \left(1-\frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \\
&2m(r) = \left(b - \frac{(a+b)^2}{4\chi(r)}\right)\left(1-\frac{a}{\chi(r)}\right)^{-\frac{(a+b)}{2a}}, \qquad r = \chi(r)\left(1-\frac{a}{\chi(r)}\right)^{\frac{(a-b)}{2a}},
\end{aligned}
\tag{5.1}
$$

where the mass function, $m(r)$, can be interpreted as a local energy [10]. One immediately finds that $m(r)$ becomes negative when $(a+b)^2/4b > \chi > a$. Therefore, one may expect that there will be significant difference not only in the global causal structures but also in the local geometry, depending on whether a is equal to b or not. However, we will find that it is not the case and that local behaviors of null geodesics significantly depend on whether a is greater than $2b$ or not.

Due to the conformally invariant nature of null geodesics, we consider null geodesics in the spacetime with a metric, $\mathbf{g} \equiv A^{-2}\hat{\mathbf{g}}$ (see Appendix A). Let $\hat{k}^\mu = dx^\mu/dv$ be null geodesic associated with $\hat{\mathbf{g}}$ with the affine parameter, v. Hereafter, a hat denotes geometrical quantities associated with $\hat{\mathbf{g}}$, and the corresponding geometrical quatities associated with $\mathbf{g}$ are denoted without a hat. Then the null geodesics associated with $\mathbf{g}$ are given as

$$
d\lambda = A^{-2}dv, \quad k^\mu = \frac{dx^\mu}{d\lambda} = A^2\hat{k}^\mu, \qquad \to \qquad \hat{k}^\alpha\hat{\nabla}_\alpha\hat{k}^\mu = k^\alpha\nabla_\alpha k^\mu = 0. \tag{5.2}
$$

Since the local geometrical nature of the spacetime is described by the Riemann curvature, it is important to examine optical scalars of the irrotational null geodesic congruence defined by Sachs [11]. Let $\{\hat{\mathbf{E}}_{(a)} : (a = 1 \sim 4)\} = \{\hat{\mathbf{k}}, \hat{\mathbf{m}}, \hat{\mathbf{t}}, \bar{\hat{\mathbf{t}}}\}$ be a null tetrad such that

$$
\hat{k}^\alpha\hat{\nabla}_\alpha\hat{E}^\mu_{(a)} = 0, \quad \hat{g}_{\mu\nu}\hat{E}^\mu_{(a)}\hat{E}^\nu_{(b)} = -2\delta^1_{(a}\delta^2_{b)} - 2\delta^3_{(a}\delta^4_{b)}. \tag{5.3}
$$

Then the optical scalars, namely, the expansion, θ, and the complex shear, σ, are defined as

$$
\hat{\theta} = \hat{\nabla}_\mu\hat{k}_\nu\hat{t}^\mu\bar{\hat{t}}^\nu, \quad \hat{\sigma} = \hat{\nabla}_\mu\hat{k}_\nu\bar{\hat{t}}^\mu\bar{\hat{t}}^\nu. \tag{5.4}
$$

The optical scalar equations are also conformally invariant, and we define the following quantities associated with $\mathbf{g}$:

$$
\theta = A^2\hat{\theta} - k^\mu\nabla_\mu \ln A, \quad \sigma = A^2\hat{\sigma}, \quad t^\mu = A\hat{t}^\mu. \tag{5.5}
$$

Then the optical scalar equations become [11, 12]

$$
\begin{aligned}
&\frac{d\theta}{d\lambda} + \theta^2 + |\sigma|^2 = -\frac{1}{2}R_{\mu\nu}k^\mu k^\nu \equiv \mathcal{R}, \\
&\frac{d\sigma}{d\lambda} + 2\theta\sigma = -R_{\mu\alpha\nu\beta}k^\mu k^\nu\bar{t}^\alpha\bar{t}^\beta = -C_{\mu\alpha\nu\beta}k^\mu k^\nu\bar{t}^\alpha\bar{t}^\beta \equiv F,
\end{aligned}
\tag{5.6}
$$

where $C_{\mu\alpha\nu\beta}$ is the Weyl curvature.

First, we examine circular orbits of null geodesics in the Just coordinate. Without loss of generality, we consider null geodesics on the $\theta = \pi/2$ plane. Then the geodesic equation is reduced to

$$
\begin{aligned}
\dot{\chi}^2 &= E^2 - \frac{L^2}{\chi^2}\left(1-\frac{a}{\chi}\right)^{\frac{(2b-a)}{a}} \equiv E^2 - V(\chi), \\
E &= \left(1-\frac{a}{\chi}\right)^{\frac{b}{a}}\dot{t}, \quad L = \chi^2\left(1-\frac{a}{\chi}\right)^{\frac{(a-b)}{a}}\dot{\phi},
\end{aligned}
\tag{5.7}
$$

where a dot denotes a derivative with respect to the affine parameter, λ, and E and L are integration constants. The circular orbits are determined by the conditions that $\dot{\chi} = 0$ and $dV/d\chi = 0$. We find that the circular orbit is obtained as $\chi = \chi_C \equiv (a+2b)/2$ for $a < 2b$, and that, surprisingly, no circular orbit exists for $a \geq 2b$ in contrast to the case in the Schwarzschild spacetime.

Dyer [12] has obtained general solutions to the optical scalar equations (5.6) in the general static, spherically symmetric spacetime, and we summarize his results in Appendix B. In the static, spherically symmetric spacetime, one finds that the Weyl driving term, F, and the shear, σ, can be regarded as real quantities without loss of generality [12]. By defining the new optical scalars, $C_\pm$, as

$$\frac{d}{d\lambda}\ln C_\pm = \theta \pm \sigma, \tag{5.8}$$

Dyer has obtained the following equations:

$$\frac{d^2 C_\pm}{d\lambda^2} = (\mathcal{R} \pm F)C_\pm. \tag{5.9}$$

We evaluate the Ricci and Weyl driving terms, $\mathcal{R}$ and F, as

$$
\begin{aligned}
\mathcal{R} &= -\frac{(a^2-b^2)}{4\chi^4}\left(1-\frac{a}{\chi}\right)^{-2}\left\{1-\frac{h^2}{\chi^2}\left(1-\frac{a}{\chi}\right)^{\frac{(2b-a)}{a}}\right\} \\
&= -\frac{(a^2-b^2)}{4\chi^4}\left(1-\frac{a}{\chi}\right)^{-2}\left(k^1\right)^2, \\
F &= \frac{h^2}{4\chi^6}\left(1-\frac{a}{\chi}\right)^{\frac{(2b-3a)}{a}}\left\{6b\chi - (a+b)(a+2b)\right\},
\end{aligned}
\tag{5.10}
$$

where $h = L/E$ is an impact parameter, and the affine parameter, λ, is chosen such that $E = 1$. It is immediately found that the Ricci driving term, $\mathcal{R}$, vanishes at the perihelion where $k^1 = 0$. A particularly interesting case is the circular orbit on which $k^1 = 0$ identically. If F vanishes on the circular orbit, the image shape of an infinitesimal light ray congruence remains unchanged. The condition, $F = 0$, is reduced to $\chi = \chi_F \equiv (a+b)(a+2b)/(6b)$ for $a > 2b$. When $a < 2b$, F is strictly positive. Since the circular orbit exists only when $a < 2b$, the condition, $\mathcal{R} = F = 0$, is never satisfied on the circular orbit. However, the condition, $\mathcal{R} = F = 0$, can be satisfied at $\chi = \chi_F$ for the scattering orbit when the impact parameter, h, is chosen as

$$h = \frac{(a+b)(a+2b)}{6b}\left[\frac{(a-b)(a-2b)}{(a+b)(a+2b)}\right]^{\frac{(a-2b)}{2a}}, \quad \rightarrow \quad \mathcal{R}(\chi_F) = F(\chi_F) = 0. \tag{5.11}$$

That is, *the gravity effectively vanishes at* χ_F in this sense.

In summary, we have found that local behaviors of null geodesics significantly depend on whether a is greater than $2b$ or not. When $a < 2b$, the behaviors are similar to those in the Schwarzschild spacetime. That is, the circular orbit exists, and the Weyl driving term, F, is strictly positive. When $a > 2b$, on the contrary, there is no circular orbit, and F changes its sign at χ_F.

These behaviors of null geodesics can be compared with those in the Reissner-Nordström spacetime:

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{5.12}$$

The Ricci and Weyl driving terms are calculated as

$$\mathcal{R} = -\frac{Q^2 h^2}{r^4}, \quad F = \frac{3h^2}{r^4}\left(\frac{m}{r} - \frac{Q^2}{r^2}\right). \tag{5.13}$$

One immediately finds that F vanished at $r = r_F \equiv Q^2/m$. However, when $m^2 > Q^2$, r_F is inside the event horizon, $r_+ \equiv m + \sqrt{m^2 - Q^2}$. When $9m^2 > 8Q^2$, two circular orbits exist:

$$r = r_C^\pm \equiv \frac{1}{2}\left(3m \pm \sqrt{9m^2 - 8Q^2}\right). \tag{5.14}$$

Moreover, when $m^2 > Q^2$, one finds that $r_C^+ > r_+ > r_F > r_C^- > r_-$, where $r_- \equiv m - \sqrt{m^2 - Q^2}$. A particular case that $r_F = r_C^-$ occurs only when $m^2 = Q^2$, and one finds that $r_F = r_C^- = m = r_\pm$ and that $r_C^+ = 2m$.

General solutions to (5.9) with (5.10) can be obtained by Dyer's formula (Appendix B.1), however, we have only examined quantitative behaviors of the Ricci and Weyl driving terms, $\mathcal{R}$ and F. Further quantitative studies of the solutions and their astronomical applications, especially the implication to gravitational lens effects, will be discussed in the forthcoming paper.

5.2 The scalar-tensor-Weyl solution

It is difficult to study generic null geodesics in the scalar-tensor-Weyl solution. Therefore, we shall examine specific null geodesics on the $y = 0$ plane. When $y = 0$, the geodesic equation is reduced to

$$\begin{aligned} \dot{W}^2 &= E^2 - \frac{L^2}{x^2 - 1}\left(\frac{x-1}{x+1}\right)^{2\delta} \equiv E^2 - V(x), \\ E &= \frac{1}{\sigma}\left(\frac{x-1}{x+1}\right)^{\delta} \dot{t}, \quad L = (x^2 - 1)\left(\frac{x-1}{x+1}\right)^{-\delta} \dot{\phi}, \end{aligned} \tag{5.15}$$

where W is a function of x defined by

$$\frac{dx}{dW} = \left(\frac{x^2}{x^2 - 1}\right)^{\frac{\Delta^2 - 1}{2}}, \tag{5.16}$$

a dot denotes a derivative with respect to the affine parameter, λ, and E and L are integration constants. The circular orbits are determined by the conditions that $\dot{W} = 0$ and $dV/dx = 0$.

We find that these conditions are reduced to $x = x_{\mathrm{C}} \equiv 2\delta$. Since x should be larger than unity, the circular orbit exists when $\delta > 1/2$.

We evaluate the Ricci and Weyl deriving terms, $\mathcal{R}$ and F, (see Appendix B.2) and find that

$$
\begin{aligned}
\mathcal{R} &= -\frac{\Delta^2-\delta^2}{\sigma^2}\frac{1}{x^2(x^2-1)}\left(\frac{x^2}{x^2-1}\right)^{\Delta^2}\left[1-\frac{(h/\sigma)^2}{x^2-1}\left(\frac{x-1}{x+1}\right)^{2\delta}\right] \\
&= -\frac{\Delta^2-\delta^2}{(x^2-1)^2}(k^1)^2, \\
F &= \frac{1}{\sigma^2}\frac{1}{x^3(x^2-1)}\left(\frac{x^2}{x^2-1}\right)^{\Delta^2}\times \\
&\quad\times\left[\delta(\Delta^2-1)+\frac{(h/\sigma)^2}{x^2-1}\left(\frac{x-1}{x+1}\right)^{2\delta}\left\{\delta(\Delta^2-1)-(\Delta^2+2\delta^2)x+3\delta x^2\right\}\right],
\end{aligned}
\tag{5.17}
$$

where $h = L/E$ is an impact parameter. One finds that they are reduced to (5.10) when $\Delta^2 = 1$. We show $\mathcal{R}$ and F in Figure 2. We numerically find that the qualitative features of $\mathcal{R}$ and F are the same as those in the spherical case ($\Delta = 1$) and that the significant qualitative changes in $\mathcal{R}$ and F are seen depending on whether $\delta < 1/2$ or not.

6 Discussions

After summarizing the spherical exterior solutions in the scalar-tensor theories of gravity, we have derived a new series of the axisymmetric exterior solutions, which are reduced to Voorhees's solutions in general relativity. We have found that the prolate solution, namely the scalar-tensor-Weyl solution, contains the spherical exterior solution as a special case in which we take some non-trivial choice of the parameters in the solution. We also tried to find exterior solutions corresponding to the Kerr solution in general relativity and have not yet succeeded in it. Note that the generic scalar-tensor-Weyl solution is not reduced to the Schwarzschild solution in the limit, $\varphi \to 0$. Therefore, we guess that the Kerr-like exterior solution must be reduced to the Tomimatsu-Sato-like solution in the limit, $\varphi \to 0$, where, by the term, the Kerr-like exterior solution, we mean the exterior solution such that it coincides with the Kerr solution when $\varphi = 0$ and that it is reduced to the spherical exterior solution when the Kerr parameter (the angular momentum of the matter) vanishes. An example of the solution with the first property is obtained by using the metric function, ψ, corresponding to the Kerr solution. However, the resulting solution does not satisfy the second property. We think that generating methods of the exact solutions (e.g. the Newman-Janis algorithm [13]) may play an important role in obtaining further exterior solutions.

It is important to note that one have to specify an explicit form of the coupling function, $A(\varphi)$, in order to obtain the complete information, especially the global geometry, of the exterior solutions in the scalar-tensor theories. We have not given any discussion on $A(\varphi)$, and, instead, we have examined local geometrical properties of the solutions by taking account of the conformally invariant nature of null geodesics. We have evaluated the Ricci and Weyl deriving terms, $\mathcal{R}$ and F, in the scalar-tensor-Weyl solution and have found that local behaviors of null geodesics significantly depend on whether the parameter, δ, which corresponds to b/a in the spherical exterior solution with $Q = 0$, is greater than $1/2$ or not. When $\delta > 1/2$, the behaviors are similar to those in the Schwarzschild spacetime. That is, the circular orbit exists, and the Weyl driving term, F, is strictly positive. When $\delta < 1/2$, on the contrary, there is no

circular orbit, and F changes its sign at χ_F. In the later case, one may expect to observe a gravitational lensing anomaly for images of distant sources appearing near χ_F. A ratio,

$$\mathcal{C}(h) \equiv \lim_{\lambda\to\infty} \frac{C_+ - C_-}{C_+}, \tag{6.1}$$

is a measure of the image deformation. In the Schwarzschild solution, $\mathcal{C}$ is a positive, decreasing function of h. However, in the spherical exterior solution with $\delta < 1/2$, $\mathcal{C}$ may become a negative increasing function of h near χ_F. Further quantitative studies of the solutions, C_+ and C_-, and their astronomical applications, especially the implication to gravitational lens effects including their observational detectability, will be discussed in the forthcoming paper.

A Summary of the scalar-tensor theories

We shall consider the simplest scalar-tensor theory [2-5,14]. In this theory, gravitational interactions are mediated by a tensor field, $\hat{g}_{\mu\nu}$, and a scalar field, $\hat{\phi}$. Hereafter, a symbol, $\hat{}$, denotes quantities or derivatives associated with $\hat{g}_{\mu\nu}$. An action of the theory is the following:

$$S = \frac{1}{16\pi}\int \left[\hat{\phi}\hat{R} - \frac{\omega(\hat{\phi})}{\hat{\phi}}\hat{g}^{\mu\nu}\hat{\phi}_{,\mu}\hat{\phi}_{,\nu}\right]\sqrt{-\hat{g}}d^4x + S_{\text{matter}}[\hat{\Psi}_{\text{m}}, \hat{g}_{\mu\nu}], \tag{A.1}$$

where $\omega(\hat{\phi})$ is a dimensionless arbitrary function of $\hat{\phi}$, $\hat{\Psi}_{\text{m}}$ represents matter fields, and S_{matter} is an action of the matter fields. The scalar field, $\hat{\phi}$, plays a role of an effective gravitational constant as $\hat{G} \sim 1/\hat{\phi}$. Varying the action by the tensor field, $\hat{g}_{\mu\nu}$, and the scalar field, $\hat{\phi}$, yields, respectively, the following field equations:

$$\hat{G}_{\mu\nu} = \frac{8\pi}{\hat{\phi}}\hat{T}_{\mu\nu} + \frac{\omega(\hat{\phi})}{\hat{\phi}^2}\left(\hat{\phi}_{,\mu}\hat{\phi}_{,\nu} - \frac{1}{2}\hat{g}_{\mu\nu}\hat{g}^{\alpha\beta}\hat{\phi}_{,\alpha}\hat{\phi}_{,\beta}\right) + \frac{1}{\hat{\phi}}(\hat{\nabla}_\mu\hat{\nabla}_\nu\hat{\phi} - \hat{g}_{\mu\nu}\hat{\Box}\hat{\phi}), \tag{A.2}$$

$$\hat{\Box}\hat{\phi} = \frac{1}{3+2\omega(\hat{\phi})}\left(8\pi\hat{T} - \frac{d\omega}{d\hat{\phi}}\hat{g}^{\alpha\beta}\hat{\phi}_{,\alpha}\hat{\phi}_{,\beta}\right). \tag{A.3}$$

Now we perform the following conformal transformation to the unphysical frame (the Einstein frame) with the metric, $\mathbf{g}$:

$$g_{\mu\nu} = A^{-2}(\varphi)\hat{g}_{\mu\nu}, \tag{A.4}$$

such that

$$GA^2(\varphi) = \frac{1}{\hat{\phi}}, \tag{A.5}$$

where G is a bare gravitational constant, and $A(\varphi)$ is referred to as a coupling function. Then the action is rewritten as

$$S = \frac{1}{16\pi G}\int (R - 2g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu})\sqrt{-g}d^4x + S_{\text{matter}}[\hat{\Psi}_{\text{m}}, A^2(\varphi)g_{\mu\nu}], \tag{A.6}$$

where the scalar field, φ, is defined by

$$\alpha^2(\varphi) \equiv \left(\frac{d\ln A(\varphi)}{d\varphi}\right)^2 = \frac{1}{3+2\omega(\hat{\phi})}. \tag{A.7}$$

Varying the action by $g_{\mu\nu}$ and φ yields, respectively,

$$\begin{aligned} G_{\mu\nu} &= 8\pi G T_{\mu\nu} + 2\left(\varphi_{,\mu}\varphi_{,\nu} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta}\right), && \text{(A.8)} \\ \Box\varphi &= -4\pi G\alpha(\varphi)T, && \text{(A.9)} \end{aligned}$$

where $T^{\mu\nu}$ represents a energy-momentum tensor with respect to $g_{\mu\nu}$ defined by

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\frac{\delta S_{\mathrm{matter}}[\hat{\Psi}_{\mathrm{m}}, A^2(\varphi)g_{\mu\nu}]}{\delta g_{\mu\nu}} = A^6(\varphi)\hat{T}^{\mu\nu}. \tag{A.10}$$

The conservation law of $T^{\mu\nu}$ is given by

$$\nabla_\nu T^\mu_\nu = \alpha(\varphi)T\nabla_\mu\varphi. \tag{A.11}$$

For the Maxwell field, one has

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right). \tag{A.12}$$

Further properties of the Maxwell field under the conformal transformation are found in Ref.[7].

In this paper, we adopt the unit, $G = 1$.

B Analytic solutions to the optical scalar equations

B.1 Static, spherically symmetric spacetime

In this Appendix, we summarize the analytic results obtained by Dyer [12]. A metric of the static, spherically symmetric spacetime is given as

$$ds^2 = -\mathrm{e}^{2C}dt^2 + \mathrm{e}^{2A}dr^2 + \mathrm{e}^{2B}d\Omega^2, \tag{B.1}$$

where A, B and C are functions of r. The null tangent vector, k^μ, to a null geodesic in this spacetime is obtained as

$$\begin{aligned} &k^0 = \mathrm{e}^{-2C}, \quad k^2 = 0, \quad k^3 = h\mathrm{e}^{-2B}, \\ &k^1 = \pm\mathrm{e}^{-(A+C)}\sqrt{1 - h^2\mathrm{e}^{2(C-B)}}, \end{aligned} \tag{B.2}$$

where a constant, h, is an impact parameter, and we assume that the geodesic is on the $\theta = \pi/2$ plane without loss of generality. The complex null vector, t^μ, is obtained as

$$\begin{aligned} &t^0 = \frac{\mathrm{e}^{-C}}{2\sqrt{2}}\left[\frac{2}{h}S_+S_-\mathrm{e}^B + H(S_+^2 + S_-^2)\right], \quad t^2 = \frac{i}{\sqrt{2}}\mathrm{e}^{-B}, \\ &t^1 = \frac{\mathrm{e}^{-A}}{2\sqrt{2}}\left[\frac{1}{h}(S_+^2 + S_-^2)\mathrm{e}^B + 2HS_+S_-\right], \quad t^3 = \frac{Hh}{\sqrt{2}}\mathrm{e}^{-2B}, \\ &S_\pm = \sqrt{\mathrm{e}^{-C} \pm h\mathrm{e}^{-B}}, \quad H = -\frac{1}{h}\int^r \frac{B'}{S_+S_-}\mathrm{e}^{B-C}dr, \end{aligned} \tag{B.3}$$

where a prime denotes a derivative with respect to r. Then the Ricci and Weyl driving terms, $\mathcal{R}$ and F, in (5.6) are evaluated as

$$\begin{aligned}\mathcal{R}+F &= \mathrm{e}^{-2(A+C)}\left(B''+(B')^2-B'C'-B'A'\right)\\ &\quad -h^2\mathrm{e}^{-2(A+B)}\left(C''+(C')^2-C'B'-C'A'\right),\\ \mathcal{R}-F &= \mathrm{e}^{-2(A+C)}\left(B''+(B')^2-B'C'-B'A'\right)\\ &\quad -h^2\mathrm{e}^{-2(A+B)}\left(\mathrm{e}^{2(A-B)}+B''-A'B'\right).\end{aligned} \tag{B.4}$$

Since F is real, one can let σ be real without loss of generality. Dyer has introduced the new optical scalars, $C_\pm$, defined by

$$\frac{d}{d\lambda}\ln C_\pm=\theta\pm\sigma, \tag{B.5}$$

and has obtained the following equations:

$$\frac{d^2C_\pm}{d\lambda^2}=(\mathcal{R}\pm F)C_\pm. \tag{B.6}$$

Dyer has obtained general solutions to $(B.6)$ as

$$\begin{aligned}C_+ &= C_+^0\sqrt{\mathrm{e}^{2B}-h^2\mathrm{e}^{2C}}\left\{\int^r\frac{\mathrm{e}^{A+B+C}}{[\mathrm{e}^{2B}-h^2\mathrm{e}^{2C}]^{\frac{3}{2}}}dr+D_+^0\right\},\\ C_- &= C_-^0\mathrm{e}^B\sin\left(h\int^r\frac{\mathrm{e}^{A+C}}{\sqrt{\mathrm{e}^{2B}-h^2\mathrm{e}^{2C}}}dr+D_-^0\right),\end{aligned} \tag{B.7}$$

where $C_\pm^0$ and $D_\pm^0$ are integration constants.

B.2 Static, axisymmetric spacetime

A metric of the static, axisymmetric spacetime is given by

$$ds^2=-\mathrm{e}^{2\alpha}dt^2+\mathrm{e}^{2\beta}dx^2+\mathrm{e}^{2\gamma}dy^2+\mathrm{e}^{2\mu}d\phi^2, \tag{B.8}$$

where metric functions, α, β, γ and μ, are functions of x and y. The tangent vector, k^μ, to a null geodesic on the $y=0$ plate is obtained as

$$\begin{aligned}&k^0=\mathrm{e}^{-2\alpha},\quad k^2=0,\quad k^3=h\mathrm{e}^{-2\mu},\\ &k^1=\pm\mathrm{e}^{-\beta}\sqrt{\mathrm{e}^{-2\alpha}-h^2\mathrm{e}^{-2\mu}},\end{aligned} \tag{B.9}$$

where a constant, h, is an impact parameter. The complex null vector, t^μ, is obtained as

$$\begin{aligned}t^0 &= \frac{1}{2\sqrt{2}}\left[\frac{2}{h}\mathrm{e}^\mu S_+S_-+H(S_+^2+S_-^2)\right]\mathrm{e}^{-\alpha},\quad t^2=\frac{i}{\sqrt{2}}\mathrm{e}^{-\gamma},\\ t^1 &= \frac{1}{2\sqrt{2}}\left[\frac{1}{h}(S_+^2+S_-^2)\mathrm{e}^\mu+2HS_+S_-\right]\mathrm{e}^{-\beta},\quad t^3=\frac{1}{\sqrt{2}}Hh\mathrm{e}^{-2\mu},\\ S_\pm &= \sqrt{\mathrm{e}^{-\alpha}\pm h\mathrm{e}^{-\mu}},\quad H=-\frac{1}{h}\int^x\frac{\mathrm{e}^{\mu-\alpha}}{S_+S_-}\frac{\partial\gamma}{\partial x}dx.\end{aligned} \tag{B.10}$$

Figure 2: The Weyl and Ricci deriving terms, F and $\mathcal{R}$, are shown, where $(\Delta, \delta, h) = (1.5, 0.3, 2, 0)$ in (a) and $(1.5, 0.7, 2.0)$ in (b). The vertical and horizontal axes denote, respectively, the deriving terms and x.

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$$
\begin{tikzcd}
\text{Scalar-Tensor-Weyl } (\Delta, \delta) \arrow[r, "\Delta \longrightarrow \delta"] \arrow[d, "\Delta \rightarrow 1"] & \text{Weyl } (\delta) \arrow[d, "\delta \rightarrow 1"] \\
\text{Spherical Exterior Solution} (\delta = b/a) \arrow[r, "\delta \longrightarrow 1"'] & \text{Schwarzschild}
\end{tikzcd}
$$

Figure 1: A schematic picture of series of the scalar-tensor-Weyl solutions.

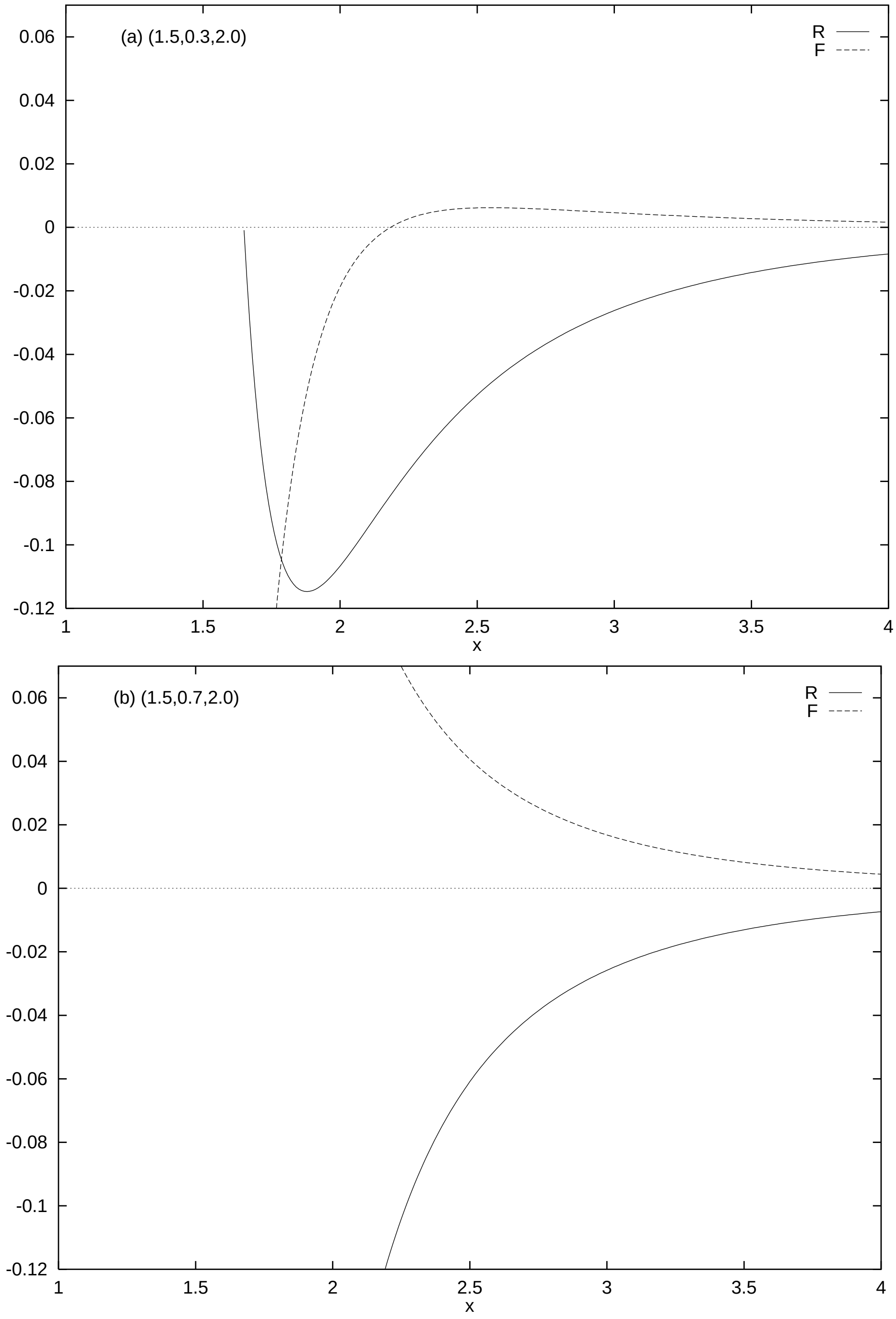

Figure 2: The Weyl and Ricci deriving terms, F and $\mathcal{R}$, are shown, where $(\Delta, \delta, h) = (1.5, 0.3, 2, 0)$ in (a) and $(1.5, 0.7, 2.0)$ in (b). The vertical and horizontal axes denote, respectively, the deriving terms and x.